PERIOD–III

M12CH10

TOPIC 10

Rigid Motion-II & Enlargement

10.1. ROTATIONAL SYMMETRY

An object is said to have a **rotational symmetry** if it can be turned around its centre to match itself in less 360° turn.

Order (or degree) of rotational symmetry

The number of times an object will fit onto itself in one complete rotation is called the order (or degree) of the rotational symmetry.

Note: The order (or degree) of rotational symmetry of a regular polygon of n sides is n. We shall see it later.

Angle of rotational symmetry

The minimum angle required for an object to rotate, either in clockwise or anti-clockwise direction, and coincide with itself is known as angle of rotational symmetry and is given by

$$\frac{360°}{\text{Order (or degree) of rotation}}.$$

Example 1. *State the rotational symmetry of the following geometrical shapes and find the angle of rotational symmetry.*

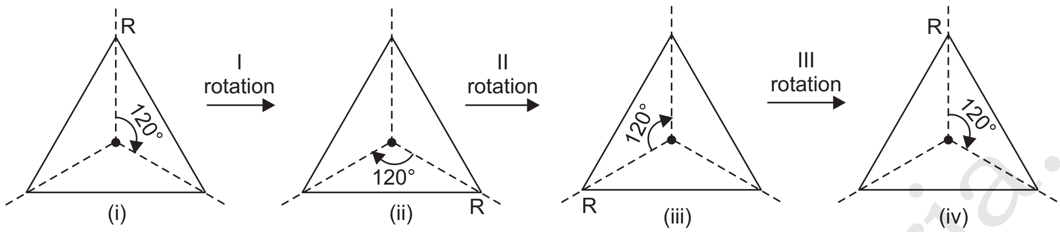
(a) *Equilateral triangle* (b) *Square*

(c) *Regular pentagon* (d) *Regular hexagon*

(e) *Rectangle*

Solution. (a) An equilateral triangle is a regular polygon of 3 sides, therefore, its order of rotational symmetry is 3 (see figure). Also,

$$\text{Angle of rotational symmetry} = \frac{360°}{3} = 120°$$

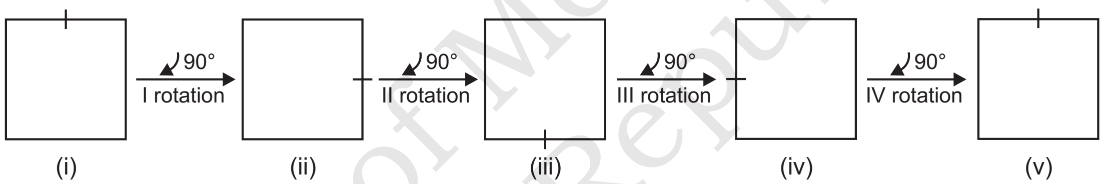


Order of rotational symmetry of an equilateral triangle

Thus, figure (i) has been rotated three times about an angle 120° in clockwise direction to get figure (iv) which is exactly same as figure (i). Hence, the order of rotational symmetry of an equilateral triangle is 3 and angle of rational symmetry is 120°.

(b) A square is also a regular polygon of 4 sides. Therefore, its order of rotational symmetry is 4 (see figure). Also,

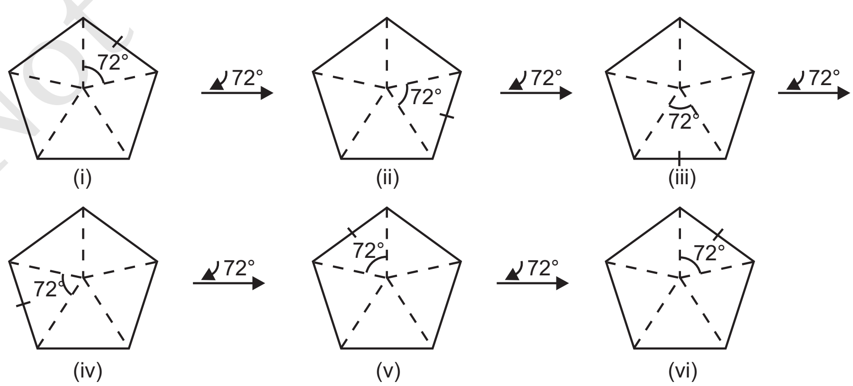
$$\text{Angle of rotational symmetry} = \frac{360^\circ}{4} = 90^\circ$$



Order of rotational symmetry of a square.

Thus, figure (i) has been rotated 4 times about an angle 90° in clockwise direction, to get figure (v) which is exactly same as figure (i). Hence, the order of rotational symmetry of a square is 4 and angle of rotational symmetry is 90°.

(c) The order of rotational symmetry for a regular pentagon (number of sides is 5) is 5 (see figure) and

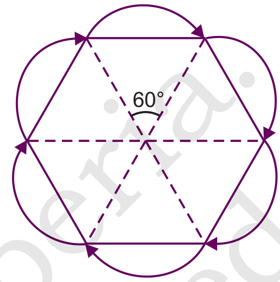


Order of rotational symmetry of a regular pentagon.

Angle of rotational symmetry = $\dfrac{360°}{5}$ = 72°.

(d) The order of rotational symmetry of a regular hexagon (number of sides is 6) is 6 (see figure). Also,

Angle of rotational symmetry = $\dfrac{360°}{6}$ = 60°.



Regular hexagon

(e) If a rectangle is rotated by 180° twice, it comes to its original position. (see figure)

∴ Order of rotational symmetry is 2.

Also, angle of rotational symmetry = $\dfrac{360°}{2}$ = 180°

Example 2. *Consider the following Ghananian coins and assume that any two vertices of coin (iii) is joined by a straight line as shown in Fig. For each coin.*

(i) (ii) (iii)

(a) *Write the geometrical name.*

(b) *Find the order of rotational symmetry.*

(c) *Find the angle of rotational symmetry.*

Solution.

(i) (a) The geometrical name is circle.

(b) Order of rotational symmetry is infinite.

(c) Angle of rotational symmetry = $\dfrac{360°}{\infty}$ = 0°.

(ii) (a) The geometrical name is regular heptagon.

(b) Order of rotational symmetry = 7

(c) Angle of rotational symmetry = $\dfrac{360°}{7}$ ≅ 51.42°.

(iii) The geometrical name is regular polygon of 12 sides.

Order of rotational symmetry = 12

Angle of rotational symmetry = $\dfrac{360°}{12}$ = 30°

10.2. ROTATION

A rotation is also a rigid motion that turns every point of a pre-image through a specified angle and direction about a fixed point.

The fixed point is called the **centre of rotation** and the amount of rotation (measured in degrees) is known as angle of rotation.

Rotation of Objects in Everyday Life

In our everyday life, we come across many objects, such as clock hands, ceiling fans, lid of a jar when the jar is opened or closed, tyres of a truck which rotate as shown in figure.

Clock hands


Ceiling fans


Lid of a jar


Wheels of a truck

When an object rotates, its shape and size do not change. Rotation takes place about a point. This point may or may not be the centre of the object.

Clockwise and anti-clockwise directions: Looking at the clock

The hands of the clock always rotate in a particular direction, which is known as **clockwise direction** and is denoted by the symbol ↻. There are many objects which can rotate in **anti-clockwise direction** (direction opposite to the hands of the clock) as well as in clockwise direction such as pinwheel, (see figure) lid of a jar, tyres of a truck etc. The anti-clockwise direction is also known as **counter-clockwise direction.**


Pinwheel

Meaning of full turn, half-turn and quarter-turn.

When an object is rotated through an angle of 360°, then, the rotation is known as **full turn rotation**.

When the angle of rotation is 180° (half of 360°), then, the rotation is known as **half turn rotation**. Similarly, a rotation through 90° ($\frac{1}{4}$ of 360°) is known as quarter rotation.

Note 2: A rotation requires centre of rotation, direction of rotation and angle of rotation.

Rotation of a point by 90° about the origin.

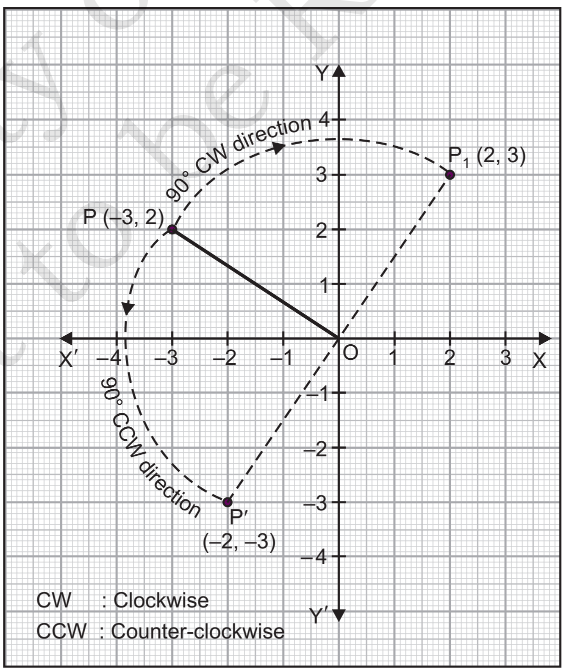
Rule: Suppose a point $P(x, y)$ is rotated through an angle of 90° **counter-clockwise** about the origin, then, the image of $P(x, y)$ after rotation is given by $P'(-y, x)$.

If the point $P(x, y)$ is rotated through an angle of 90° **clockwise** about the origin, then, the image of $P(x, y)$ after rotation is given by $P'(y, -x)$.

Example 3. *Suppose a point (–3, 2) is rotated through an angle of 90°. Find its images if the point is rotated* (i) *clockwise* (ii) *anti-clockwise. Draw the images on the graph or square paper.*

Solution. Let the point is $P(-3, 2)$.

When the point $P(-3, 2)$ is rotated by 90° anti-clockwise (or counter-clockwise) its image will be $P'(-2, -3)$.



Images of the point (–3, 2)

When the point $P(-3, 2)$ is rotated by 90° clockwise, its image will be $P_1(2, 3)$.

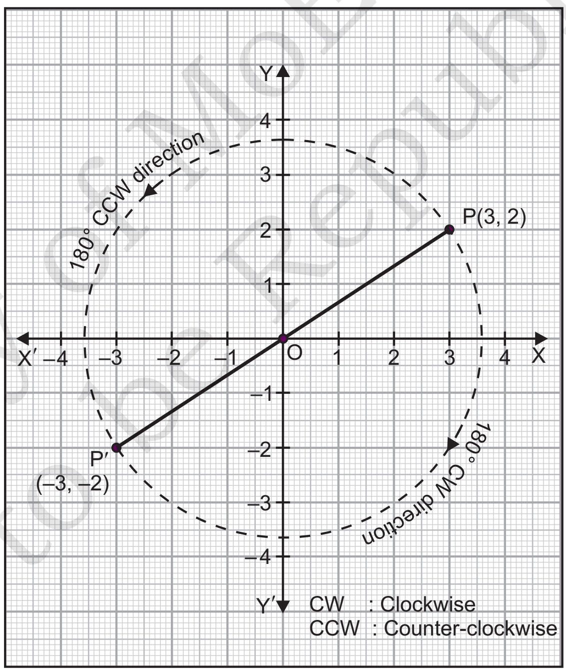
The images are shown on the graph paper (see figure).

Rotation of a point by 180° about the origin

Rule: Suppose a point $P(x, y)$ is rotated through an angle of 180° in clockwise or counter-clockwise direction, then, its image will be $P'(-x, -y)$. Thus, a rotation through 180° needs **no direction**.

Example 4. *Suppose a point P(–3, 2) is rotated through an angle of 180°, either clockwise or counter-clockwise. Find and draw its image point on the graph or square paper.*

Solution. A rotation through 180° needs no direction. The image of $P(3, 2)$ is the point $P'(-3, -2)$ and is shown in Fig.



Images of the point (3, 2)

Rotation of a point by 270° about the origin

Rule: Suppose a point $P(x, y)$ is rotated through an angle of 270° counter-clockwise, then, its image will be $P'(y, -x)$, which is same if the point $P(x, y)$ is rotated by 90° clockwise. Hence CCW rotation by 270° = CW rotation by 90°

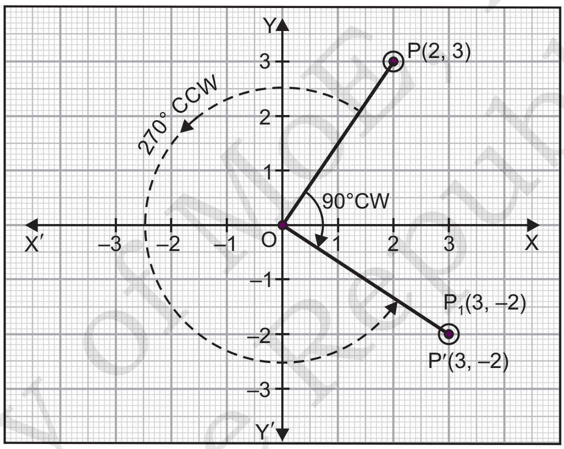
Similarly, if the point $P(x, y)$ is rotated through an angle of 270° clockwise, then, its image will be $P_1(-y, x)$, which is same if the point $P(x, y)$ is rotated by 90° counter-clockwise. Hence,

CW rotation through 270° = CCW rotation through 90°.

Example 5. *Find and draw the image of the point P(2, 3) if it is rotated by 270° counter-clockwise. Also, draw the image of P(2, 3) if it is rotated by 90° clockwise. Are the images same?*

Solution. If the point $P(2, 3)$ is rotated by 270° counter-clockwise, its image will be $P'(3, -2)$, which is shown on the graph paper (see figure).

If the point $P(2, 3)$ is rotated by 90° clockwise, its image will be $P_1(3, -2)$, which is same as $P'(3, -2)$ (figure).



Images of the point (2, 3)

Rotation of a point by 360° about the origin

Rule: Suppose a point $P(x, y)$ is rotated through an angle of 360° either in clockwise or counter-clockwise direction, then, the image of $P(x, y)$ will be the same point $P(x, y)$.

Rotation of plane figures about the origin

The procedure is explained in the following examples.

Example 6. *Graph the image of Fig. when it is rotated by*

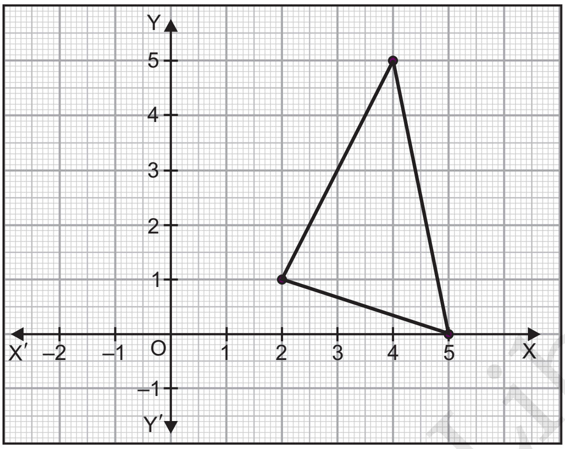
(a) *90° counter-clockwise about the origin.*

(b) *90° clockwise about the origin.*

(c) *270°clockwise about the origin.*

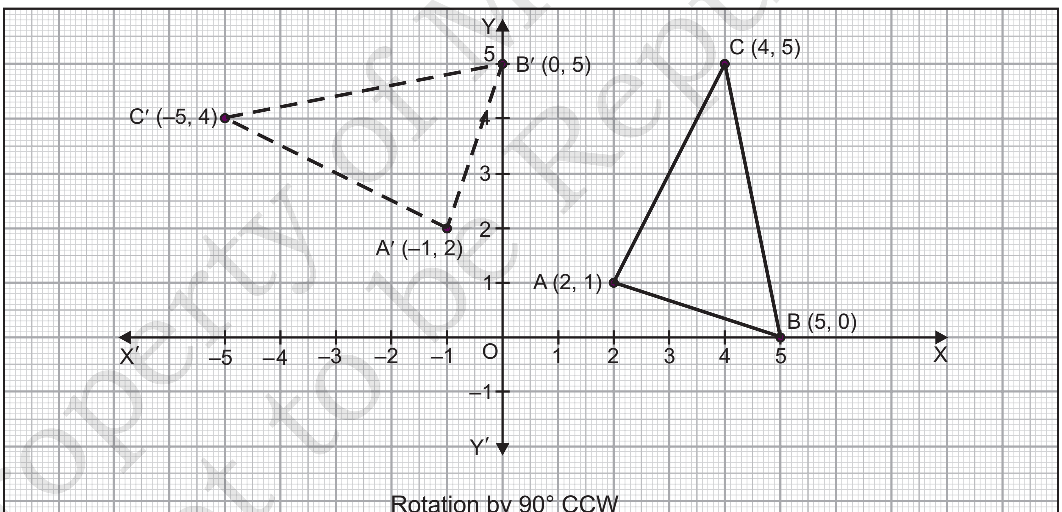
(d) *270° counter-clockwise about the origin.*

Also, write the vertices of the image triangle in each case.



Solution. Let the vertices of the given triangle are $A(2, 1)$, $B(5, 0)$ and $C(4, 5)$ respectively.

(a) If the triangle ABC is rotated by 90° counter-clockwise, then, the image triangle $A'B'C'$ is shown in the following graph (see figure).



Rotation by 90° CCW

Point	Image
A (2, 1)	A′ (–1, 2)
B(5, 0)	B′(0, 5)
C(4, 5)	C′(–5, 4)

Image of triangle *ABC* after rotating by 90° counter-clockwise.

(b) If the triangle ABC is rotated by 90° clockwise then, the image triangle $A_1B_1C_1$ is shown in the following graph (see figure).

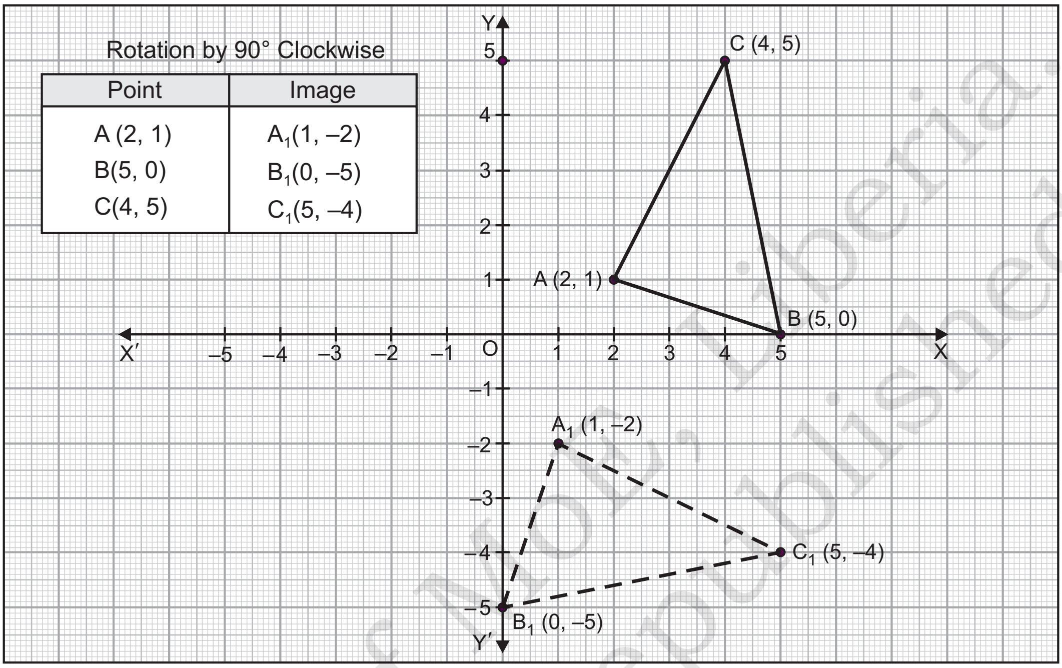


Image of triangle *ABC* after rotating by 90° clockwise

(c) Since, rotation through 270° clockwise

= rotation through 90° counter clockwise,

∴ The image triangle will be same as in part (a)

(d) Since rotation through 270° counter clockwise

= rotation through 90° clockwise,

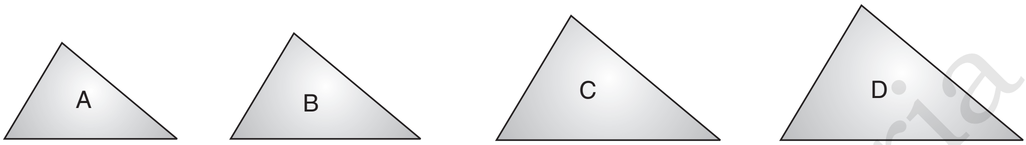
∴ The image triangle will be same as in part (b).

10.3. ENLARGEMENTS

In our previous discussion, we have learnt about three main transformations: translation, reflection and rotation. The other important transformation is enlargement.

Drawing enlargement of geometrical figures with given scale factors

Consider the following figure:



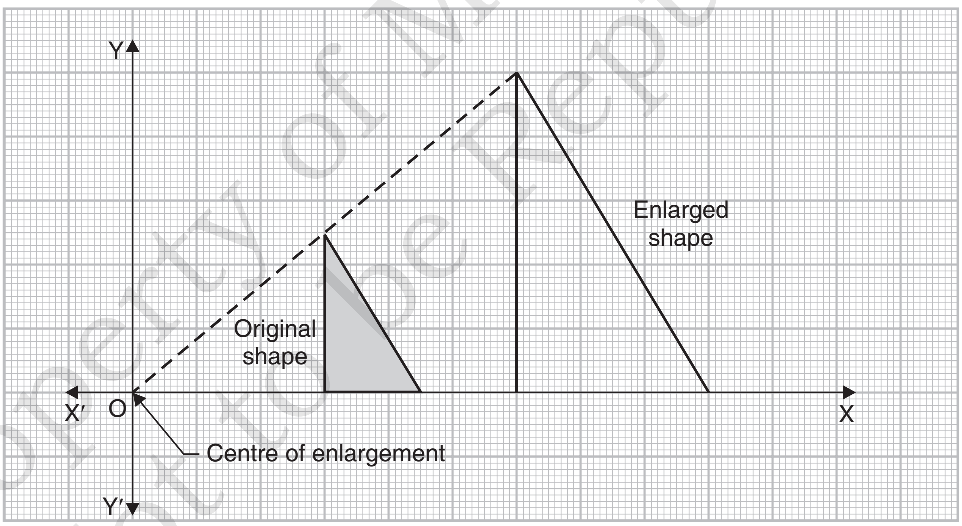
Shapes *B*, *C* and *D* are all enlargements of shape *A*. However, they are all in different positions and of different sizes.

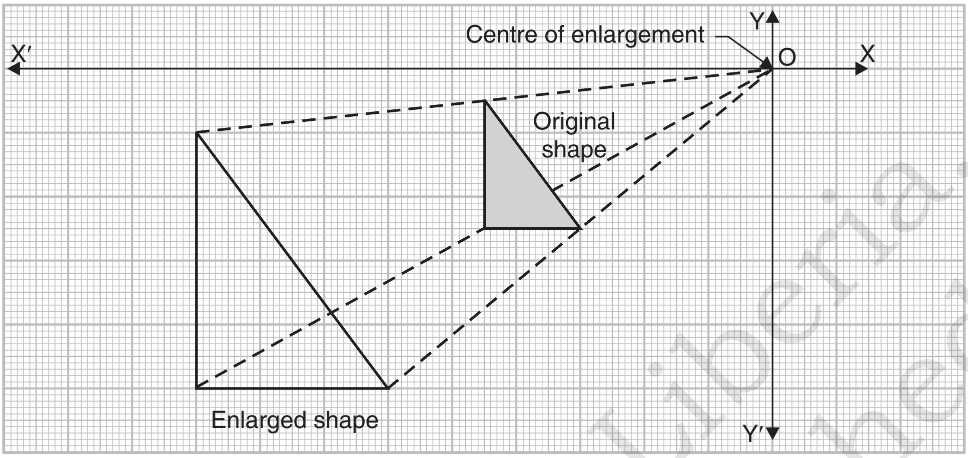
When drawing the enlargement of a geometrical figure, we must state their positions and their sizes.

Centre of Enlargement

The position of the enlarged shape (figure) is described by the centre of enlargement.

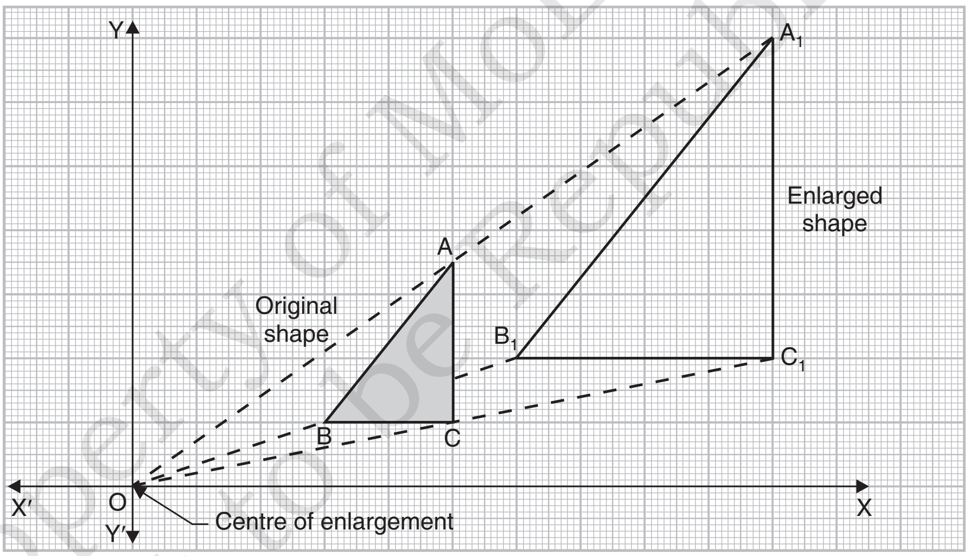
In the following enlargements, [see figure (a–b)] point *O* is the centre of enlargement.





Scale factor

The size of an enlargement is described by its scale factor.



For example, a scale factor of 2 means that the new shape (figure) is twice the size of the original. A scale factor of 3 means that the new shape is three times the size of the original. Triangle $A_1B_1C_1$ (see figure) is an enlargement of triangle ABC with a scale factor of 2.

What do you observe?

For a scale factor of 2,

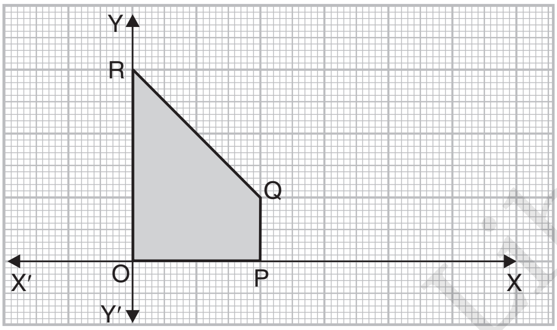
$OA_1 = 2 \times OA$, $OB_1 = 2 \times OB$, $OC_1 = 2 \times OC$

Points A_1, B_1 and C_1 are connected to form an enlarged triangle $A_1B_1C_1$.

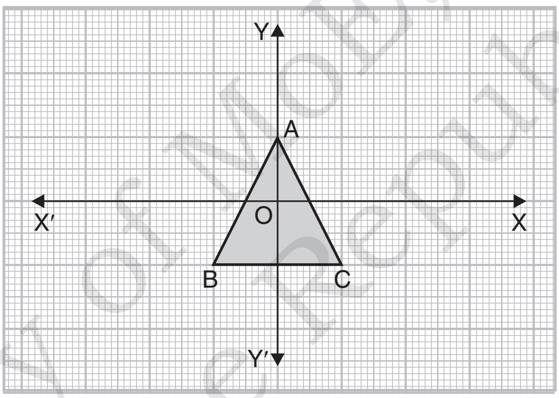
Note: In an enlargement there is a centre of enlargement and a scale factor.

Example 7. *Draw the enlargements of the following figures (figure (a) and figure (b)) with O as a centre of enlargement*

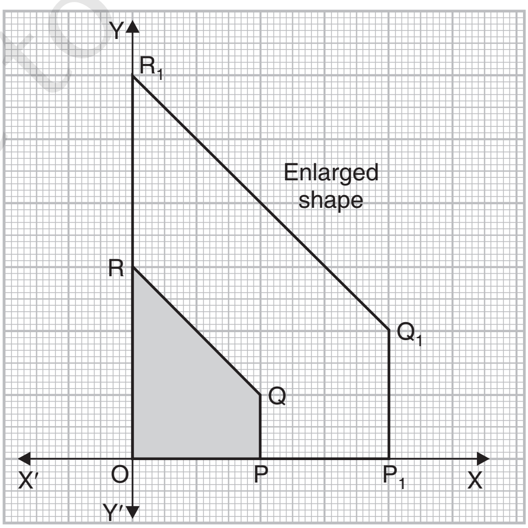
(a) *Use a scale factor of 2.*



(b) *Use a scale factor of 3.*



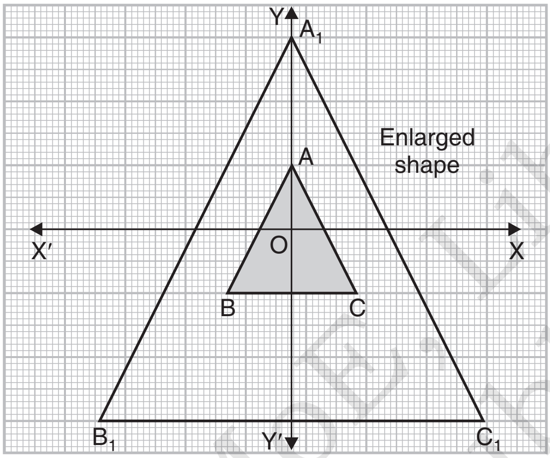
Solution. (a) The enlargement $OP_1Q_1R_1$ of the given shape $OPQR$ with O as a centre of enlargement, using a scale factor of 2 is shown in figure.



For a scale factor of 2,

$$OP_1 = 2 \times OP,\ OQ_1 = 2 \times OQ,\ OR_1 = 2 \times OR$$

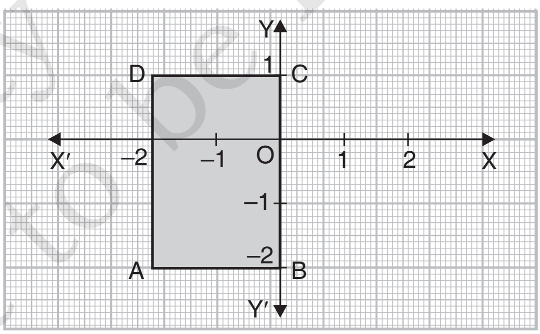
(b) The enlargement $A_1B_1C_1$ of the given shape ABC with O as a centre of enlargement, using a scale factor of 3 is shown in figure.



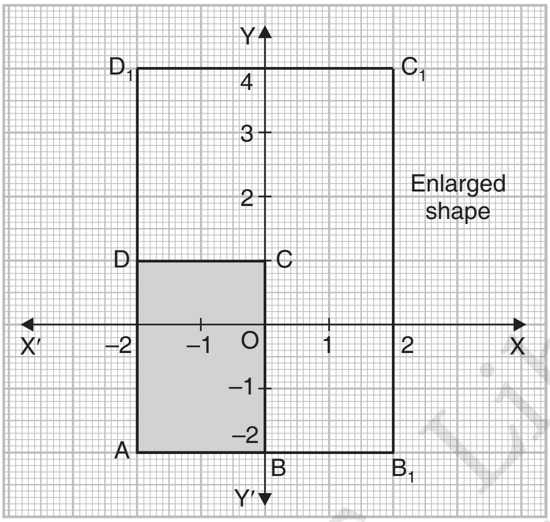
For a scale factor of 3,

$$OA_1 = 3OA,\ OB_1 = 3OB,\ OC_1 = 3OC.$$

Example 8. *Draw an enlargement of rectangle ABCD (Fig. 51) with the centre of enlargement A and a scale factor of 2.*



Solution. The enlargement $AB_1C_1D_1$ of the given shape $ABCD$ with A as the centre of enlargement and a scale factor of 2 is shown in figure.



For a scale factor of 2,

$$AB_1 = 2AB,\ AC_1 = 2AC,\ AD_1 = 2AD$$

Stating single transformation that maps one shape onto another

Look carefully at the following figure:

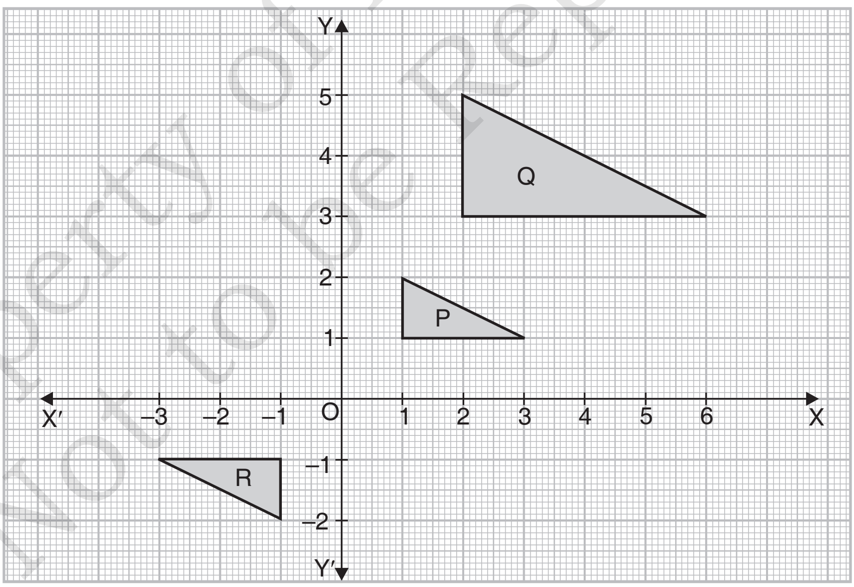


Fig. 53

There is a single transformation that:

(i) maps triangle P onto triangle Q. (see figure)

In the figure, the single movement or transformation that takes the point (1, 1) of triangle P to the point (2, 3) of triangle Q is translated by the vector $\begin{pmatrix}1\\2\end{pmatrix}$ and using a scale factor of 2.

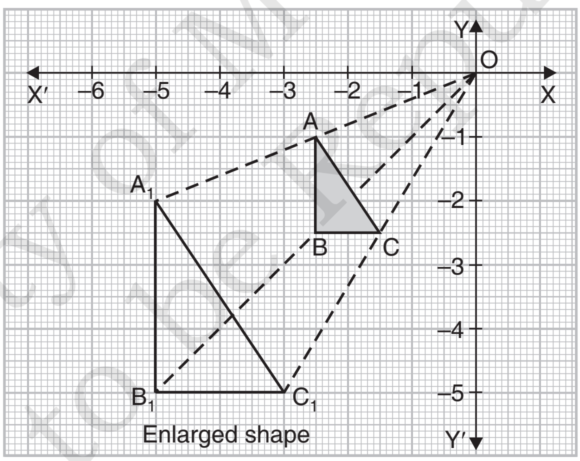
(ii) maps triangle P onto triangle R (see figure).

In the figure, the single transformation that maps triangle P onto triangle R is $(x, y) \to (-x, -y)$.

Finding the Scale Factor

In this section, we will learn about finding the scale factor by which an object is enlarged. The scale factor indicates by how much an object has been enlarged.

What is the scale factor of the following enlargement (see figure)?



The scale factor of the enlargement is 2.

Let us measure the sides of the object ABC.

Here, AB = 1.5 cm, BC = 1 cm, AC = 1.8 cm

Also the measure of the sides of the image $A_1B_1C_1$ are

$$A_1B_1 = 3 \text{ cm}, B_1C_1 = 2 \text{ cm}, A_1C_1 = 3.6 \text{ cm}$$

Now, $\dfrac{A_1B_1}{AB} = \dfrac{B_1C_1}{BC} = \dfrac{A_1C_1}{AC} = 2.$

So, the ratio of corresponding sides is equal to the scale factor.

What do you conclude?

To find the scale factor of an enlargement, we determine the ratio of the sides of an image to the corresponding sides of the object. This ratio is the required scale factor.

Properties of Enlargement

Enlargement is also called dialation, contraction, compression, or even expansion. The enlarged shape becomes bigger or smaller.

Characteristics of Enlargements

An enlargement is a transformation that produces an image that has the *same shape* as the original, but has a *different* size. It stretches or shrinks the original figure. As discussed earlier, the description of an enlargement includes the centre of enlargement and a scale factor.

The centre of enlargement is a fixed point in the plane about which all the points are expanded or contracted. It is the only invariant point under an enlargement.

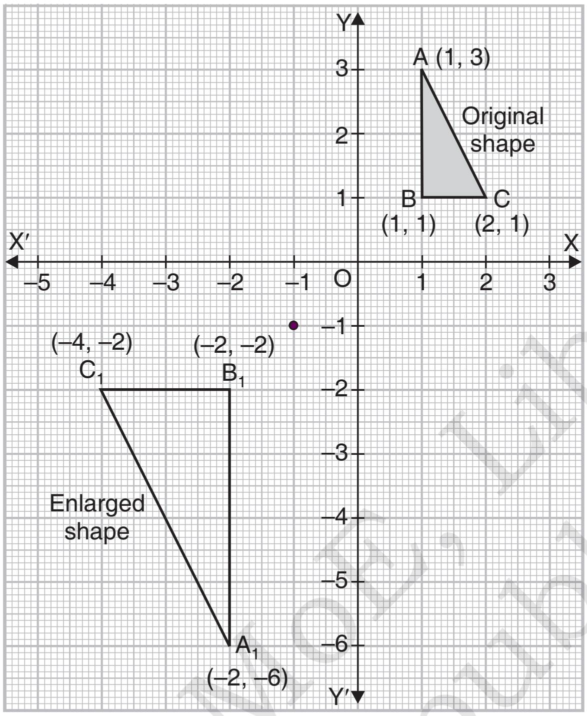
The scale factor connecting the object with its image is usually denoted by the letter K and calculated by

$$K = \frac{\text{Length of the image side}}{\text{Length of the object side}}$$

Let us investigate the characteristics of enlargements under the following conditions of the scale factor:

- ***If the scale factor (K) is negative.***

The image and the figure are on the opposite sides of the centre of enlargement. For example, the enlargement of triangle ABC with the centre of enlargement at the origin and a scale factor of –2 is shown in figure.



Observe that every coordinate of the original triangle has been multiplied by the scale factor –2.

- ***If the scale factor (K) is greater than 1 or less than –1.***

If $K > 1$, then the image is larger than the object and on the same side of the centre of enlargement (Figs.).

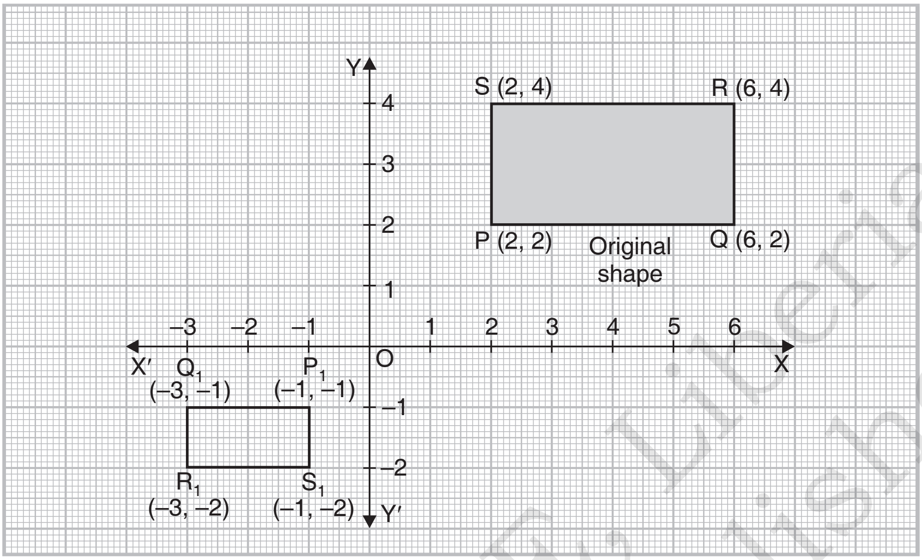
If $K < -1$, then the image is larger than the object and on the opposite side of the centre of enlargement (see figure).

- ***If the scale factor (K) is between –1 and 1 (i.e., a fraction)***

If $-1 < K < 0$, then the image is smaller than the object and on the opposite side of the centre of enlargement (see figure).

The following figure shows the enlargement of rectangle *PQRS* with the centre of enlargement at *O* and the scale factor of $-\frac{1}{2}$.

Observe that every coordinate of the original rectangle has been multiplied by the scale factor of $-\frac{1}{2}$.



If 0 < K < 1, then the image is smaller than the object.

Fig. shows the enlargement of a pentagon *OABCD* with the centre of enlargement at *O* and the scale factor of $\frac{1}{3}$.

Observe that every coordinate of the original pentagon *OABCD* has been multiplied by the scale factor $\frac{1}{3}$ to get the pentagon $OA_1B_1C_1D_1$.

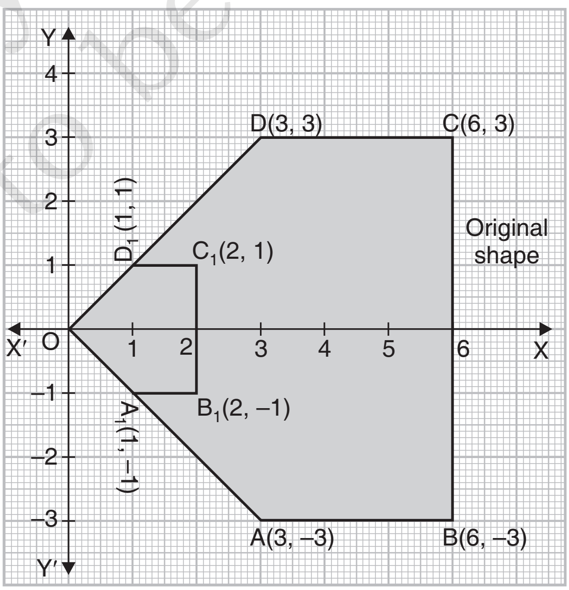


Fig. 58

10.4. SCALE DRAWING

There are many objects in the world which are too big or small for a normal sheet of paper. But we can draw these to a particular "*scale*" relative to these objects.

A *scale drawing* is a reduction or an enlargement of a plane figure that is similar to the original figure.

An example from real life: Since it is not always possible to draw on paper the actual size of real-life objects such as the real size of a car, an airplane, we need scale drawings to represent their sizes.

This is the picture of a van (Fig.).

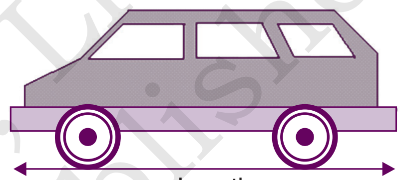


Fig.

In real-life, the length of this van may measure 6 m or 600 cm. However, the length of a copy or print paper that you could use to draw this van is a little bit less than 30 cm. Since 600 ÷ 30 = 20, you will need about 20 sheets of copy paper to draw the length of the actual size of the van.

In order to use just one sheet, you could then use 1 cm on your drawing to represent 20 cm on the real-life object.

You can write this situation as 1 : 20 or $\frac{1}{20}$ or 1 to 20.

Notice that the first number always refers to the length of the drawing on paper and the second number refers to the length of real life object.

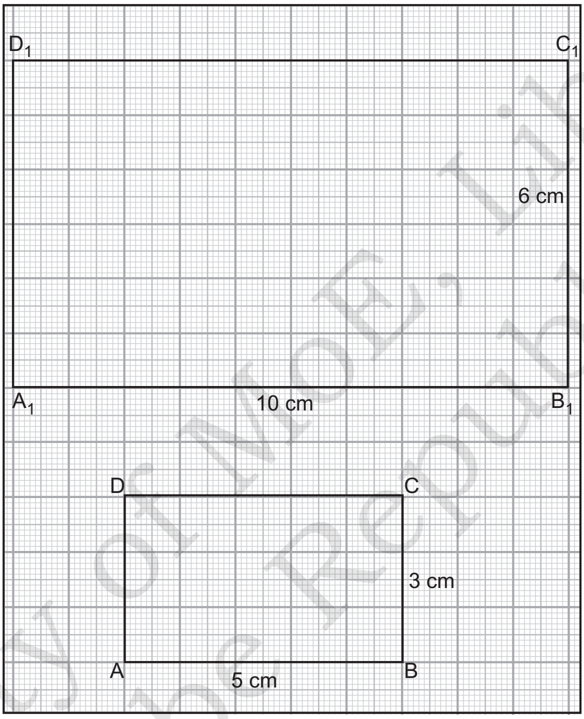
We summarize the above discussion as below.

1. A scale is the ratio of the length in a drawing (or model) to the length of the real object.

2. A **scale drawing** is a drawing that shows a real objet with accurate sizes except they have been reduced or enlarged using scale.

3. In a scale drawing, all dimensions are reduced or enlarged by the same proportion.

Scale factor: A scale factor K is simply a non-zero number that multiplies the dimensions of an object. This can make an object larger or smaller depending upon the values of K. If $K > 1$, then, multiplying the dimensions of an object by K will increase the size of the object. If $K < 1$, it will decrease the size. If $K = 1$, then the two objects are similar or congruent. Consider a rectangle $ABCD$ (Fig.) with its width as 3 cm and length as 5 cm. Now, multiplying the dimensions of this rectangle by 2, the new width is 6 cm = 2 × 3 cm and length is 10 cm = 2 × 5 cm.



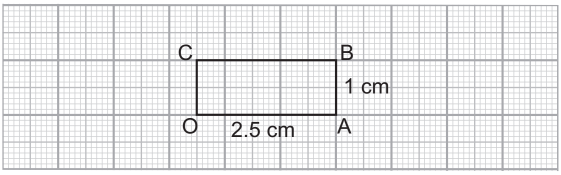
∴ The new rectangle $A_1B_1C_1D_1$ is shown in Fig., which has same shape, but not of same size. Here, the number 2 is a scale factor.

Thus, dimensions of the resulting figure = K × dimensions of the given figure, where K is the scale factor.

Note: A scale factor has no unit.

Scale factor and areas and volumes of plane figures. Given the scale factor, we can determine areas and volumes of plane figures as explained in the following examples.

Example 9. *Draw the enlargement of the following rectangle (Fig.) by using a scale of 1 : 2. Hence, find the area of the resulting rectangle.*



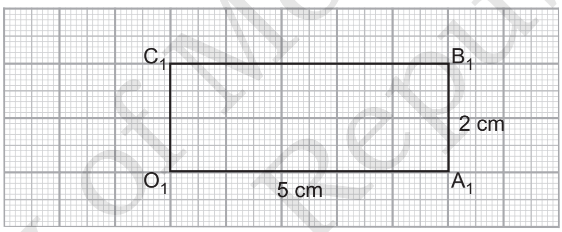
Solution. Looking at the graph, the dimensions of the rectangle are $OA = 2.5$ cm, $AB = 1$ cm.

Here the scale is 1 : 2 (from smaller rectangle to larger rectangle), therefore, scale factor = $\frac{2}{1} = 2$.

$\therefore$ The new dimensions are

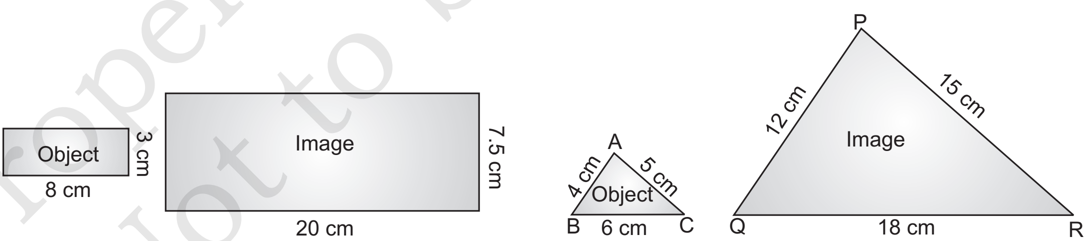
$$O_1A_1 = 2 \times 2.5 = 5 \text{ cm and}$$
$$A_1B_1 = 2 \times 1 = 2 \text{ cm}$$

The enlarged rectangle $O_1A_1B_1C_1$ is shown in Fig.



$\therefore$ Area of $O_1A_1\ B_1C_1 = 5 \times 2 = 10$ cm^2

Example 10. *Find the scale factor of the following reductions (Fig. 68).*



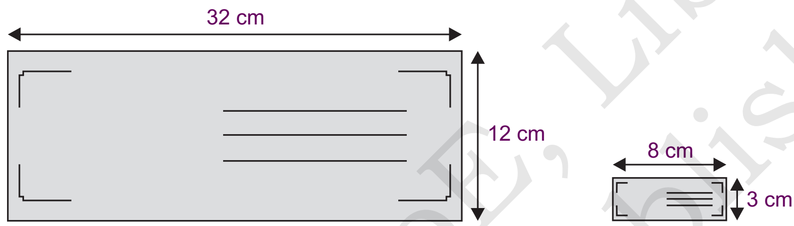
(a) (b)

Solution. (a) We are required to find the scale factor from the larger figure (image) to the smaller figure (object).

$$\therefore \quad \text{Scale factor} = \frac{\text{Ratio of one side of object}}{\text{Ratio of the corresponding side of image}} = \frac{8}{20} = \frac{2}{5}$$

(b) Scale factor $= \dfrac{6}{18} = \dfrac{1}{3}$.

Example 11. *A design for an invitation has been scaled down using a photocopier. The original and scaled copy measurements are shown below (see figure). Find the scale factor.*



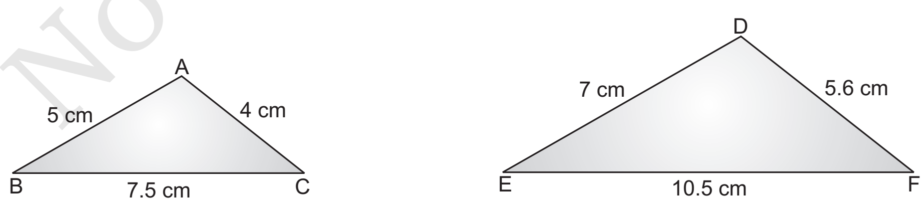
Solution. The design of the invitation has been scaled down (from larger to smaller)

$$\therefore \quad \text{Scale factor} = \frac{8}{32} = \frac{1}{4}.$$

10.5. AREAS AND VOLUMES OF SIMILAR FIGURES

"The scale factor from the first figure to the second figure is always the ratio of any one side of the second figure to the corresponding side of the first figure".

For example, the two similar triangles (Fig.) ΔABC and ΔDEF have a scale factor of $\dfrac{10.5}{7.5} = 1.4$ from the smaller triangle to the larger triangle.



Remember that if we write

(i) Actual dimensions of the object = K × dimension of the model, then, K will be greater than one.

(ii) Dimension of the model = K × actual dimension of the object, then, K will be less than one.

Scale Factor and Area: Consider two rectangles of dimensions 5 cm × 3 cm and 10 cm × 6 cm respectively. The scale factor is 2. Why?

Area of first rectangle = $5 \times 3 = 15\ \text{cm}^2$

Area of second rectangle = $10 \times 6 = 60\ \text{cm}^2 = 4 \times 15\ \text{cm}^2$

$= 2^2 \times$ Area of first rectangle

Thus, we can say that

Area of the resulting figure = K^2 × Area of the given figure, where K is the scale factor.

Scale Factor and Volume: Consider a rectangular prism of dimensions 4 cm × 2 cm × 3 cm. If the scale factor is 2, the new dimensions of the rectangular prism will be 8 cm × 4 cm × 6 cm. Why?

Volume of first rectangular prism = $4 \times 2 \times 3 = 24\ \text{cm}^3$

Volume of second rectangular prism

$= 8 \times 4 \times 6 = 192\ \text{cm}^3 = 8 \times 24\ \text{cm}^3$

$= 2^3 \times$ Volume of first rectangular prism

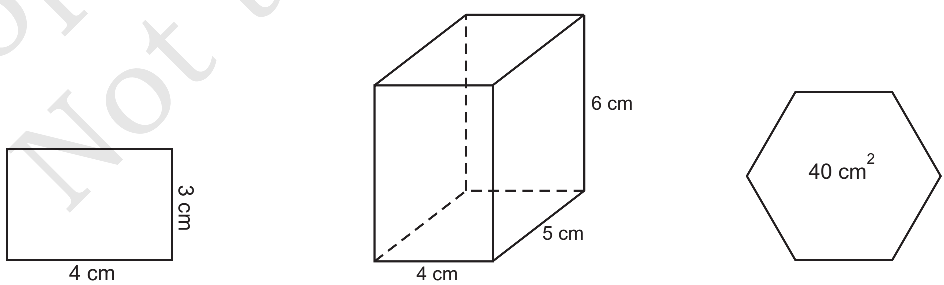
Thus, we can say that

Volume of the resulting figure

$= K^3$ × Volume of the given figure,

where K is the scale factor

Example 12. *Consider the following geometrical shapes (Fig. 76).*



(a) *What will be the area of the rectangle if it is enlarged with a scale factor 3?*

(b) *What will be the volume of the cuboid if it enlarged with a scale factor 3?*

(c) *What will be the area of the hexagon if it enlarged with a scale factor 4?*

Solution. (a) Area of the given rectangle = $4 \times 3 = 12$ cm^2

$\therefore$ Scale factor $K = 3$

$\therefore$ Area of the resulting rectangle = $3^2 \times 12 = 108$ cm^2

(b) Volume of the given cuboid

$= 4 \times 5 \times 6 = 120$ cm^3

Scale factor $K = 3$

$\therefore$ Volume of the resulting figure = $3^3 \times 120 = 3240$ cm^3

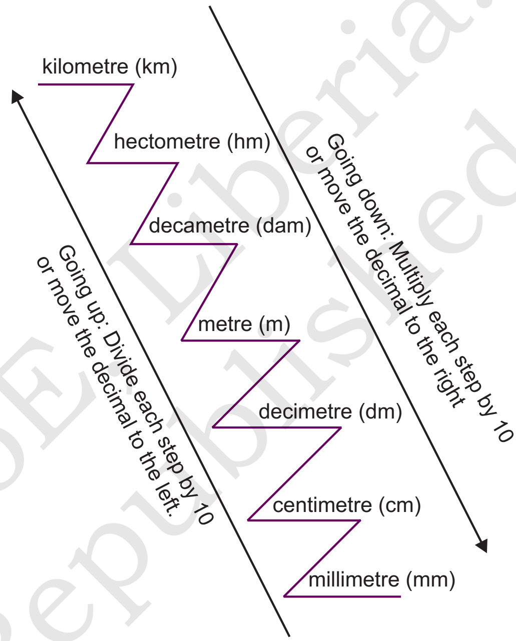
(c) Area of the given hexagon

$= 40$ cm^2

Scale factor $K = 3$

$\therefore$ Area of the resulting hexagon = $3^2 \times 40 = 360$ cm^2

We choose $K = 2 > 1$ since K is the scale factor for enlarged square.



Conversion of units.

Example 13. *A triangle with area 18 cm^2 is reduced with a scale factor $\frac{1}{2}$, find the area of reduced triangle.*

Solution. Area of reduced triangle

$= K^2 \times$ area of the given triangle

$$= \frac{1}{4} \times 18 = 4.5 \text{ cm}^2$$

EXERCISE

1. Draw and name any one Adinkra symbol that has

(*i*) rotational symmetry of order 0

(*ii*) rotational symmetry of order 2

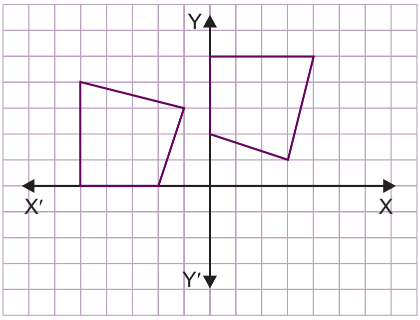
(*iii*) rotational symmetry of order 4

(*iv*) *rotational symmetry of order 12*

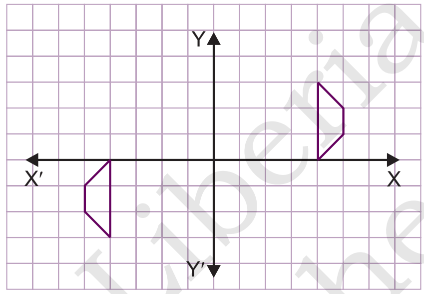
Also, find the angle of rotational symmetry and number of mirror lines in each case.

2. Below are given some geometrical shapes and their images (Fig.). Write a rule to describe each rotation.

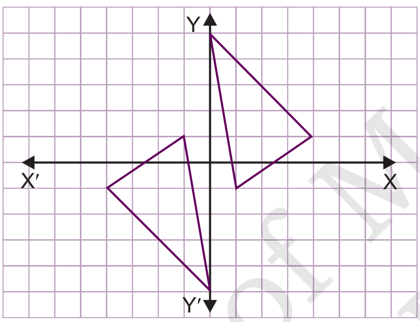
(a)



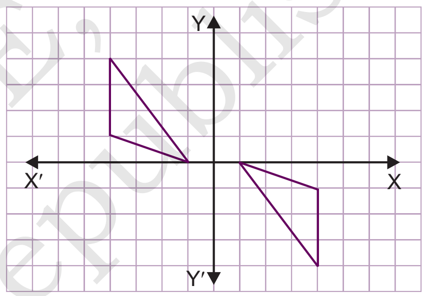
(b)



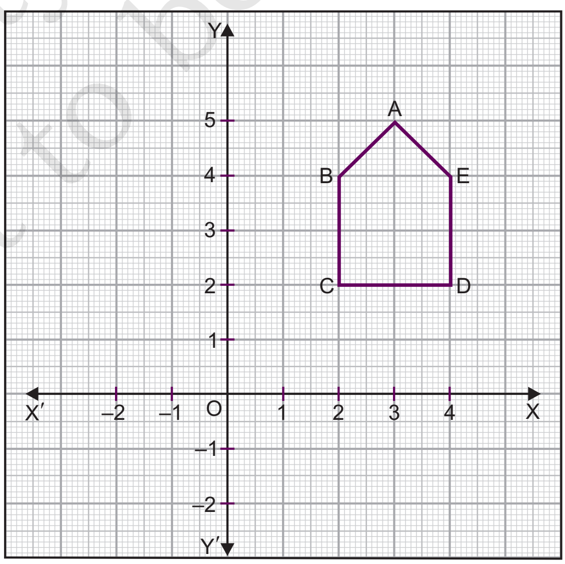
(c)



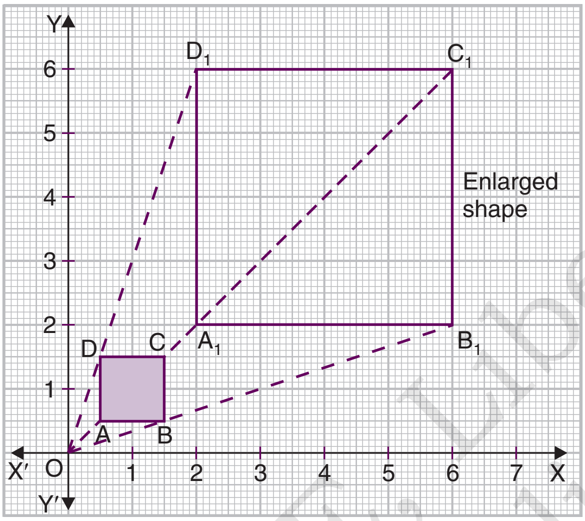
(d)



3. Rotate the pentagon ABCDE as shown in Fig. by 90° clockwise about the vertex E. Also draw the image figure.



4. Find the scale factor of the following enlargement (Fig.).



5. A cone (Figure) with diameter 12 cm and height 76 cm is to be reduced by a scalar factor of $\frac{1}{4}$. What will be the dimensions of the reduced cone? Also, find the volume of the reduced cone.